Deterministic implementation of weak quantum cubic nonlinearity

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We propose a deterministic implementation of weak cubic nonlinearity, which is a basic building block of a full-scale continuous-variable quantum computation. Our proposal relies on preparation of a specific ancillary state and transferring its nonlinear properties onto the desired target by means of deterministic Gaussian operations and feed forward. We show that, despite the imperfections arising from the deterministic nature of the operation, the weak quantum nonlinearity can be implemented and verified with the current level of technology.

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I. INTRODUCTION

Ever since it has been first mentioned by Feynman [1], quantum computation has been the holy grail of quantum-information theory, because the exponential speedup it offers promises to tackle certain computational problems much faster than any classical computer could [2]. The original approach to quantum computing relied on manipulation of discrete quantum systems [3], but later it was shown that the same speedup can be achieved by computing with continuous-variable (CV) quantum systems, and that CV systems may be even more effective [4,5].

Besides the readily available operations with Hamiltonians composed of first (linear) and second (quadratic) powers of quadratic operators $\hat{x}$ and $\hat{p}$, CV quantum computation requires a single kind of nontrivial resource—a single operation with a Hamiltonian at least cubic (third power) in quadrature operators [4]. Unfortunately, the currently achievable experimental interaction strengths are too low compared to noise to be of use.

Fortunately, the need for currently unavailable cubic unitary evolution may not be so dire. Let us recall the original statement of Lloyd and Braunstein [4]: If one has access to Hamiltonians $\hat{A}$ and $\hat{B}$, one can approximatively implement an operation with Hamiltonian $i[\hat{A},\hat{B}]$. *Approximatively* is the key term here, meaning that the desired operation is engineered only as a quadratic polynomial of the interaction time:

$$e^{i\hat{A}t}e^{i\hat{B}t}e^{-i\hat{A}t}e^{-i\hat{B}t} \approx e^{-[\hat{A},\hat{B}]t^2} + O(t^3). \tag{1}$$

Consequently, even the initial operations need not be unitary—their quadratic approximations are fully sufficient. What this means is that if we take interest in a sample cubic interaction with Hamiltonian $\hat{H} \propto \hat{x}^3$, we need not implement the unitary $e^{i\chi\hat{x}^3}$, where χ is a real parameter, but it is enough to be able to perform operation

$$\mathcal{O}_6(\hat{x}) = 1 + i\chi\hat{x}^3 - \chi^2\hat{x}^6/2. \tag{2}$$

This is the lowest order expansion for which the commutator trick (1) works, but let us start with the real lowest order expansion, $1 + \beta\hat{x}^3$, where β is a complex number. This expansion behaves as a weak cubic coupling if β is imaginary and has the added benefit that it can be used to compose (2) when the respective values of β are complex and chosen properly. In principle, even this gate can be further decomposed into series of $1 + \gamma\hat{x}$ ($\gamma \in \mathbb{C}$) operations [6]. These phase sensitive gates can be implemented probabilistically on a traveling beam of light by subsequent application of photon subtraction and photon addition, represented by operators $\hat{a} = (\hat{x} + i\hat{p})/\sqrt{2}$ and $\hat{a}^\dagger$ [7–9]. They are very useful for preparing various ancillary states, but for use in a full-fledged information processing we are interested in their *deterministic* implementation.

II. IDEAL IMPLEMENTATION

To this end we employ the approach of [10], thoroughly discussed in [11], where it was suggested that a unitary operation acting on a state can be deterministically implemented with the help of a proper resource state, a quantum nondemolition (QND) coupling, a suitable measurement, and a feed-forward loop. Explicitly, for operation $\mathcal{O}(\hat{x})$ acting on the pure state $|\psi\rangle = \int \psi(x)|x\rangle dx$, the resource state is $\mathcal{O}(\hat{x})|p = 0\rangle$. After QND coupling, represented by the unitary $\hat{U}_{\mathrm{QND}}(\lambda) = e^{i\lambda\hat{x}_2\hat{p}_1}$, is employed and the overall state is transformed to

$$\int \psi(x)\mathcal{O}(y)|y - \lambda x, x\rangle dx dy, \tag{3}$$

the ancillary resource mode gets measured by a homodyne detection. We can for now assume $\lambda = 1$, as the overall message remains unchanged. For any detected value q the output state is

$$\int \psi(x)\mathcal{O}(x + q)|x\rangle dx. \tag{4}$$

To obtain the desired result, one either postselects only for situations when $q = 0$ was detected, or applies a feed forward which would compensate for $x + q$ in the argument of the operator. It has been shown in [10] that if the desired operation $\mathcal{O}(x)$ is a unitary operation driven by a Hamiltonian of order n, the feed-forward operation requires a Hamiltonian of order $n - 1$. Explicitly, imperfections in the operator $\mathcal{O}(\hat{x} + q) = \exp[i\chi(\hat{x} + q)^3]$ can be compensated by the unitary operator $\hat{U}_{\mathrm{FF}} = \exp[-i\chi(3q\hat{x}^2 + 3q^2\hat{x})]$, which is a combination of displacement, squeezing, and phase shifts. The operation (2) we are interested in is not unitary, but since it is an

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approximation of a unitary driven by a cubic Hamiltonian, a feed forward of squeezing and displacements should perform adequately, up to some error. We'll get to this issue later. In fact, the operations available for feed forward limit us in what we can do. With squeezing and displacement we can implement only cubic operations. Of course, with them we could also tackle Hamiltonians of the fourth order, and so on. And there is another limitation—since the feed forward must be deterministic and noiseless, and therefore unitary, it can be only used to deterministically compensate unitary (at least approximatively) operations whose Hamiltonian is Hermitian. Therefore we cannot use the trick of implementing a series of $1+\gamma\hat{x}$ operations; we have to implement operation (2) in one go. Consequently, we need a sufficiently complex resource state.

III. RESOURCE STATE GENERATION

Let us now shift our attention to the required resource state. In realistic, even if idealized, considerations, one has to, instead of a position eigenstate, use a squeezed state $S|0\rangle = [\int \exp(-x^2/g)|x\rangle dx]/(\pi g)^{1/4}$, which approaches the ideal form as $g\to\infty$. The resource state can now be expressed as $\mathcal{O}(\hat{x})\hat{S}|0\rangle = \hat{S}\mathcal{O}(\hat{x}/\sqrt{g})|0\rangle$ which is a state finite in a Fock basis with a superficial layer of squeezing. As it has a finite structure, the state can be engineered by a sequence of six photon additions [12] or photon subtractions [13]. This is an extremely challenging task; let us therefore first focus at the lowest nontrivial cubic Hamiltonian expansion, $\mathcal{O}_3(\hat{x}) = 1+\chi x^3$, which is a feasible extension of recent experimental works [14]. The appropriate resource state looks like

$$\hat{S}(1+\chi'\hat{x}^3)|0\rangle = \hat{S}\left(|0\rangle + \chi'\frac{3}{2\sqrt{2}}|1\rangle + \chi'\frac{\sqrt{3}}{2}|3\rangle\right), \quad (5)$$

with $\chi' = \chi g^{-3/2}$. This state can be generated from a squeezed state by a proper sequence of photon subtractions and displacements [13], which acts as $(\hat{a}-\alpha)(\hat{a}-\beta)(\hat{a}-\gamma)\hat{S}|0\rangle$. Since the squeezing operation transforms the annihilation operator as $\hat{S}^\dagger\hat{a}\hat{S} = \mu\hat{a} - \nu\hat{a}^\dagger$, where $\mu = \cosh(\ln\sqrt{g})$ and $\nu = \sinh(\ln\sqrt{g})$, the required displacements can be obtained as a solution of the set of equations:

$$\begin{gathered} A = \alpha\beta\gamma, \quad \alpha+\beta+\gamma = 0, \\ 2\sqrt{2}\nu^3 = A\chi', \quad 3\nu^2 + 3\mu\nu = (\alpha\beta+\alpha\gamma+\beta\gamma), \end{gathered} \quad (6)$$

where A is a constant parameter related to normalization. The solution exists and it can be found analytically as

$$\begin{aligned} \alpha &= \frac{\xi+\sqrt{\xi^2-4\zeta}}{2}, \\ \beta &= \frac{\xi-\sqrt{\xi^2-4\zeta}}{2}, \\ \gamma &= -(\alpha+\beta). \end{aligned} \quad (7)$$

Here ξ and ζ are solutions of the set of equations

$$xy + C_1 = 0, \quad y - x^2 - C_2 = 0, \quad (8)$$

where $C_1 = \nu^3 2\sqrt{2}\chi'^{-1}$ and $C_2 = 3\nu^2 + 3\mu\nu$. The solutions of (8) always exist and they can be obtained analytically using the Cardan formula. The squeezing used in the state generation can be in general different from the squeezing in (5). However, squeezing can be considered to be a well accessible operation, and we shall therefore not deal with this in detail. It should be noted that an alternative way of preparing the state (5) lies in performing a suitable projection onto a single mode of a two-mode squeezed vacuum state. Engineering of the proper measurement, which too requires three avalanche photodiodes (APDs) and three displacements, leads to similar equations as in the previous case (6) with the solution of the same form.

IV. REALISTIC IMPLEMENTATION

With the resource state at our disposal we can now look more closely at the two ways to implement the gate, the probabilistic and the deterministic, in order to compare them and see what is the manifestation of high order nonlinearity in the deterministic case. The probabilistic implementation is rather straightforward. Using the resource state (5) we are able to transform the initial state to

$$|\psi_0\rangle = \int \psi(x)\mathcal{O}(x)e^{-x^2/2g}|x\rangle dx, \quad (9)$$

and as the squeezing of the resource state approaches infinity, the produced state approaches its ideal form. The final state is always pure and the actual composition of the operator $\mathcal{O}_n(x)$ can be arbitrary, allowing us, for example, to implement the operator $\mathcal{O}$ in n different nonunitary steps. On the other hand, if the resource squeezing is insufficient compared to the distribution of the state in phase space, it seriously affects some properties of the state—for example, moments of x quadrature may not be preserved any more.

But let us move toward the more interesting part, the deterministic approach. In this case the operation produces a mixed state

$$\rho' = \int P(q)|\psi_q\rangle\langle\psi_q|dq. \quad (10)$$

Here, $P(q)$ represents the probability of measuring a specific outcome q, and

$$\begin{aligned} |\psi_q\rangle = \frac{1}{\sqrt{P(q)\mathcal{N}_R}} \int \psi(x)e^{-(x+q)^2/2g}\mathcal{O}(x+q) \\ \times e^{-i\chi q^3 - i3\chi(xq^2+x^2q)}|x\rangle dx, \end{aligned} \quad (11)$$

where $\mathcal{N}_R$ is the norm of the resource state, stands for the respective quantum state corrected by feed forward. Ideally, $\mathcal{O}(x+q)e^{-i3\chi(xq^2+x^2q)} \approx \mathcal{O}(x)$, but this relation can obviously work only when both x and q are small enough for the exponent to be reasonably approximated by the finite expansion $\mathcal{O}_n$. It is, therefore, quite unfortunate that the very condition required for the operation to work flawlessly, the need for $g\to\infty$, is compatible with the feed forward only in the limit of $\chi\to 0$. To quantify these properties in greater detail we need to employ a suitable figure of merit.

V. ANALYSIS

To evaluate the quality of the approximate operation is not a straightforward task. If we want to conclusively distinguish the cubic type nonlinear interaction from a Gaussian one, we can take advantage of the known way the quadrature operators transform: $\hat{x} \rightarrow \hat{x}$, $\hat{p} \rightarrow \hat{p} + \chi \hat{x}^2$. If we apply the operation, in form of a black box, to a set of known states, we can analyze the transformed states to see whether the operation could be implemented by a suitable Gaussian, or if it is more of what we aim for. The analysis can be as easy as checking the first two moments of the quadrature operators, because the nonlinear dependence of $\langle \hat{p} \rangle$ on $\langle \hat{x} \rangle$ can not be obtained by a Gaussian operation, unless we consider a rather elaborate detection-and-feed-forward setup, which would, however, introduce an extra noise detectable either by checking the purity of the state, or by analyzing higher moments $\langle \hat{x}^2 \rangle$ and $\langle \hat{p}^2 \rangle$.

The case with a purity of 1 is straightforward to verify—as soon as the first moments have the desired form, $\langle \hat{x}' \rangle = \langle \hat{x} \rangle$ and $\langle \hat{p}' \rangle = \langle \hat{p} \rangle + \chi \langle \hat{x}^2 \rangle$, we can be certain a form of the desired non-Gaussianity is at play. In the presence of noise, the confirmation process is more involved, and we shall deal with it in a greater detail.

It needs to be shown that, in comparison to the deterministic approximation, no Gaussian operation can provide the same values of moments $\langle \hat{x}' \rangle$, $\langle \hat{x}'^2 \rangle$, and $\langle \hat{p}' \rangle$ without also resulting in a significantly larger value of moment $\langle \hat{p}'^2 \rangle$ caused by the extra noise. The complete Gaussian scheme consists of an arbitrary Gaussian interaction of the target system with a set of ancillary modes followed by a set of measurements of these modes yielding values which are used in a suitable feed forward to finalize the operation. In the case where the approximate transformations approach the ideal scenario, i.e., when $\langle \hat{x}' \rangle = \langle \hat{x} \rangle$, $\langle \hat{x}'^2 \rangle = \langle \hat{x}^2 \rangle$, and $\langle \hat{p}' \rangle = \langle \hat{p} \rangle + \chi \langle \hat{x}^2 \rangle$, only a single ancillary mode is sufficient, the optimal Gaussian interaction is in the QND interaction with a parameter λ, and after a value of ξ is measured by a homodyne detection, the feed-forward displacement of $\kappa \xi^2$ ensures the correct form of the three moments. In the end, the Gaussian approximated state can be expressed as

$$\rho_S'' = \int dx \, \hat{D}_S(\kappa x^2)_A \langle x | \hat{U}_{\mathrm{QND}}(\lambda) \hat{\rho}_S \otimes |0\rangle\langle 0| \hat{U}_{\mathrm{QND}}^{\dagger}(\lambda) |x\rangle \hat{D}_S^{\dagger}(\kappa x^2), \tag{12}$$

where the subscripts S and A denote the signal and the ancillary mode, respectively. The high order classical nonlinearity is induced by the nonlinear feed forward, represented by the displacement $\hat{D}(\alpha)$. The strength of the QND interaction λ remains a free parameter over which can the procedure be optimized to obtain the best approximation characterized by the minimal possible value of the extra noise term in $\langle \hat{p}'^2 \rangle$.

We analyze the aforementioned properties over a set of small coherent states α with $|\alpha| < 2$. We compare the state obtained by the approximate cubic interaction with the state created by the Gaussian method. In principle, this could be done for both the deterministic and the probabilistic approach, but since the probabilistic approach has the potential to work perfectly, we shall keep to deterministic methods in our comparative endeavors. For each coherent state and its cubic-gate transformed counterpart, we can, from knowledge of the first moments of quadrature operators, estimate the actual cubic nonlinear parameter and use it to construct the benchmark Gaussian-like state (12). The final step is

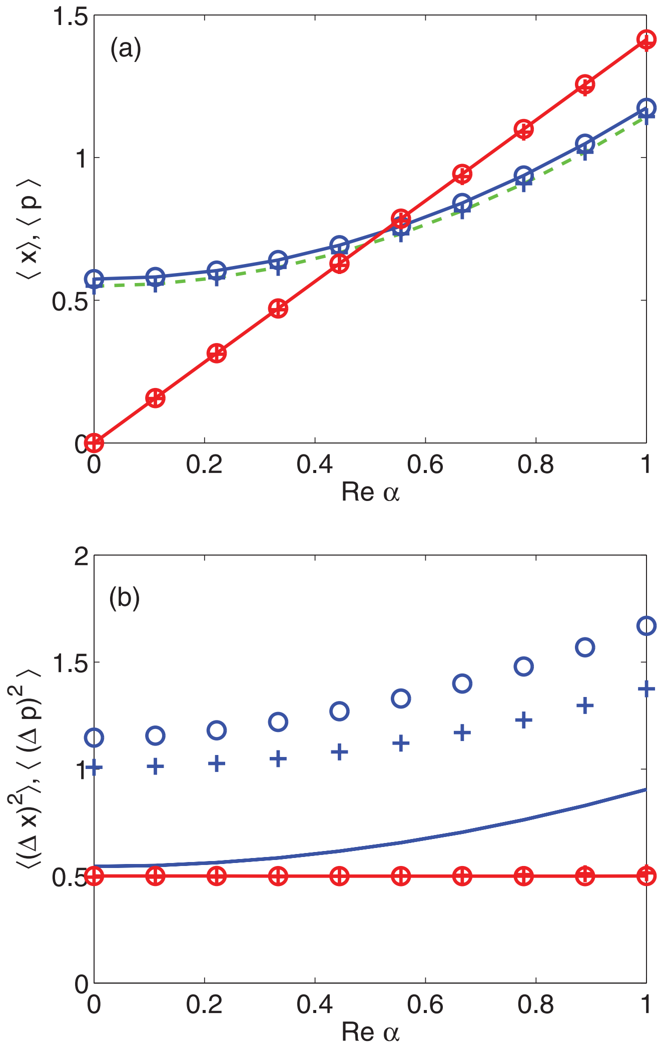

FIG. 1. (Color online) (a) First moments relative to the real part of α. Solid red and blue lines represent the ideal values of $\langle \hat{p} \rangle$ and $\langle \hat{x} \rangle$, respectively. Red and blue crosses then show these values for the deterministic non-Gaussian approximation, while red and blue circles do so for the Gaussian approximation. Dashed green line is a quadratic fit for $\langle \hat{p} \rangle$. The experimental parameters are $g = 1$ and $\chi = 0.03$. (b) Second moments relative to the real part of α. Solid red and blue lines represent the ideal values of $\langle \hat{p}^2 \rangle$ and $\langle \hat{x}^2 \rangle$, respectively. Red and blue crosses then show these values for the deterministic non-Gaussian approximation, while red and blue circles do so for the Gaussian approximation.

to compare the extra noise present in $\hat{p}$ quadrature—if the added noise for the approximate state is below the Gaussian benchmark, we can assume a non-Gaussian nature of the operation.

As an example, let us look at a particular scenario, in which the deterministic cubic gate was applied to a set of coherent states with the imaginary part of the complex amplitude constant. The effect of the operation is illustrated in Fig. 1. Figure 1(a) shows the first moments and reveals that for this purpose, effective cubic nonlinearity of $\chi_{\mathrm{eff}} = 0.1$ can be reliably obtained for both the non-Gaussian and the Gaussian approaches. Differences arise, though, for the second moments, where the value of the Gaussian quadrature moment

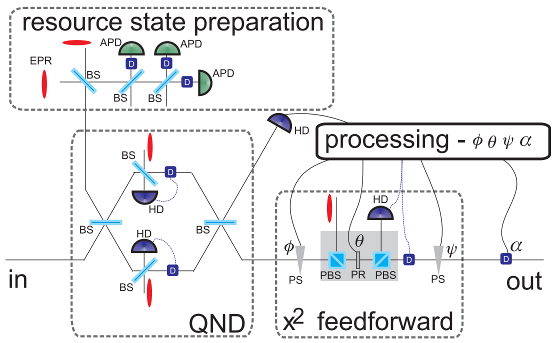

FIG. 2. (Color online) Schematic experimental setup of the deterministic x^3 gate. BS, beam splitter; PBS, polarization beam splitter; PS, phase shift; PR, polarization rotator; APD, avalanche photodiode; HD, homodyne detection; and D, displacement.

$\langle \hat{p}^2 \rangle$ is observably higher than the value of its non-Gaussian counterpart. The values of the Gaussian moment were obtained by optimizations of (12) for each particular value of Reα; it is, therefore, a stronger benchmark than a universal Gaussian operation, working over the whole range of Reα, would be. And it is still beaten by the imperfect deterministic non-Gaussian method with no squeezing in the ancillary mode.

VI. EXPERIMENTAL SETUP

The resource state generation requires three-photon subtraction from a squeezed or a two-mode squeezed light, with appropriate displacements (Fig. 2). Two photon subtractions have been already implemented [14] and three of them are within reach. The resource state is then coupled with the input using a QND gate with offline squeezing [15,16], which can be modified as to reliably manipulate the non-Gaussian resource state [17]. The final step lies in performing a sequence of feed forwards driven by a homodyne measurement of the ancilla [10]. Of those, the only nontrivial one is given by the unitary $e^{i\lambda x^2}$, where the actual value of λ depends on the measurement. This operation can be decomposed into a sequence of a phase shift by ϕ_1, squeezing with gain g_f, and another phase shift by ϕ_2, where the parameters satisfy $\tan\phi_1 \tan\phi_2 = -1$, $\tan\phi_1 = g_f$, and $(1 - g_f^4)\cos\phi_1 \sin\phi_2 = 2g_f\lambda$. Adjusting the squeezing gain on the fly can be done by exploiting the universal squeezer [15,18], where the amount of squeezing is controlled by changing the ratio of the beam splitter, which can be done by a sequence of a polarization beam splitter, the polarization rotator, and another polarization beam splitter, where the rotator controls the splitting ratio. The nonlinear dependence of the feed-forward parameters on the measurement results requires a sufficiently fast data processing, but that too is available today [19].

VII. CONCLUSION

We have proposed an experimentally feasible way of deterministically achieving weak nonlinearity of the third order. The procedure effectively engineers the operation on a single photon level and then deterministically cuts and pastes the properties onto the target state. This is reminiscent of the teleportation based gates presented in [20,21], but there are a few crucial differences. In the teleportation based gates, there is only a single resource state for both the teleportation and for the imparting of the nonlinear properties. As such, the state needs to be highly entangled, because otherwise the state would be transferred with too much noise to be of any further use. The need for a high entanglement then clashes with the limited-photon-number nature of the nonlinearity. In our implementation, the ancillary resource state has no squeezing at all, which allows the imparted nonlinearity to be observably large, while the Gaussian mediating interaction is driven by strong squeezing, ensuring minimal noise added during the operation. The limited number of photons still plays a role, though, and the nonlinearity can be faithfully applied only to target states which are sufficiently weak. Furthermore, since there is no such thing as a free lunch, the subsequent use of the transformed state in attempts to generate higher nonlinearities as per [4] requires higher and higher numbers of single photons used in the engineering.

The approach is not flawless. There are several sources of noise which can be simultaneously reduced only in the limit of an infinitely small (read unobservable) interaction. This is due to the finite photon approximation of the cubic gate not being unitary and therefore not perfectly correctable by the unitary feed forward. Nevertheless, we have shown that even with this noise, a demonstration of decisively non-Gaussian high order quantum deterministic nonlinearity going well beyond classical attempts, based on higher order nonlinearity in the feed-forward loop, can be observed already now.

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